

# GENUS ZERO BPS INVARIANTS FOR LOCAL $\mathbb{P}^1$

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**ABSTRACT.** We study the equivariant version of genus zero BPS invariants of the total space of rank 2 bundle on  $\mathbb{P}^1$  whose determinant is  $\mathcal{O}_{\mathbb{P}^1}(-2)$  by means of the moduli space of stable sheaves of dimension one as proposed by Sheldon Katz [10]. We count the torus fixed stable sheaves of low degrees and show the results agree with the prediction in local Gromov-Witten theory studied in [3].

## 1. INTRODUCTION

The 0-pointed genus  $g$  Gromov-Witten invariant for a Calabi-Yau threefold  $X$  in the curve class  $\beta \in H_2(X, \mathbb{Z})$  is defined as the degree of the virtual cycle of the moduli space of stable maps to  $X$ .

$$N_\beta^g(X) := \deg[\overline{M}_{g,0}(X, \beta)]^{\text{vir}}.$$

By the BPS state counts in M-theory, Gopakumar and Vafa [5] proposed an integer-valued invariants  $n_\beta^g(X)$  of  $X$ , called the *BPS invariants*, which are related to the Gromov-Witten invariants by the Gopakumar-Vafa formula

$$\sum_{\beta, g} N_\beta^g(X) q^\beta \lambda^{2g-2} = \sum_{\beta, g, k} n_\beta^g(X) \frac{1}{k} \left( 2 \sin \left( \frac{k\lambda}{2} \right)^{2g-2} q^{k\beta} \right).$$

A priori, the BPS invariants defined by above formula are rational numbers because the Gromov-Witten invariants are rational numbers. The *integrality conjecture* is an assertion that they are integers.

The genus zero part of above formula is

$$N_\beta^0(X) = \sum_{m|d} \frac{n_{\beta/m}^0(X)}{m^3}. \quad (1)$$

Sheldon Katz [10] proposed a mathematical definition for the genus zero BPS invariants. He considered the Donaldson-Thomas type invariants of the moduli space of stable sheaves of dimension one. He showed (1) holds for an embedded contractible rational curves. Shortly thereafter, Jun Li and Baosen Wu [13] studied K3 fibred local Calabi-Yau threefolds and proved (1).

In [2], Jim Bryan and Amin Gholampour studied the equivariant version of BPS invariant for the resolution of ADE polyhedral singularities  $\mathbb{C}^3/G$ . As the moduli space of sheaves is non-compact, the virtual cycle is not well-defined. But using a natural  $\mathbb{C}^*$ -action induced from an action on  $\mathbb{C}^3/G$ , they defined the BPS invariants via equivariant residue integrals of the virtual cycle at the fixed locus. In this paper, we follow this approach and study the equivariant version of BPS invariants for local  $\mathbb{P}^1$ .

Let  $X$  be the total space of rank two vector bundle

$$E \simeq \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-2-k)$$

on  $\mathbb{P}^1$ . Since  $\det E \simeq K_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$ ,  $X$  is a non-compact Calabi-Yau threefold in the sense that its canonical bundle is trivial.

The Gromov-Witten theory of  $X$  is studied by Jim Bryan and Rahul Pandharipande [3]. They used the natural  $(\mathbb{C}^*)^2$ -action on  $X$  via scalar multiplication on each fiber, and computed residue Gromov-Witten invariants by localization and degeneration methods. After taking anti-diagonal subtorus of  $(\mathbb{C}^*)^2$ , they got a closed formula for the Gromov-Witten partition function [3, Cor. 7.2].

We use a torus action for which the torus also acts nontrivially on the base curve  $\mathbb{P}^1$ . By Calabi-Yau condition, our action restricts to the action of their antidiagonal subtorus (Section 2). So, we can expect the genus 0 Gopakumar-Vafa formula (1) holds for the total space  $X$  of  $E$ .

**Conjecture 1.1.** *For  $\beta = d[\mathbb{P}^1] \in H_2(X, \mathbb{Z})$ , let  $N_d^{\text{GW}}(k)$  be the genus 0 local Gromov-Witten invariant computed in [3] and  $n_d(k)$  be the equivariant local BPS invariant defined by the residue integral in Definition 2.1. Then, the Gopakumar-Vafa formula*

$$N_d^{\text{GW}}(k) = \sum_{m|d} \frac{n_{d/m}(k)}{m^3}$$

*holds.*

We prove this conjecture for  $d = 1, 2$  and 3 for any  $k$  and for  $d = 4$  and  $k \leq 100$ . We show the moduli space of stable sheaves on  $X$  is smooth, and count the torus fixed sheaves using the classification of equivariant sheaves by Kool [12].

*Acknowledgements.* I would like to thank my advisor Sheldon Katz for invaluable discussions and many suggestions for improvements. I would also like to thank Martijn Kool for kindly explaining his work to me.

## 2. LOCAL BPS INVARIANT

Let  $k$  be an integer with  $k \geq -1$ . Let  $X = \text{Spec}(\text{Sym}(E^\vee))$  be the total space of a rank 2 bundle

$$E = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-2-k)$$

on  $\mathbb{P}^1$ . As a toric variety,  $X$  contains a torus  $T' = (\mathbb{C}^*)^3$  and has two  $T'$ -invariant affine open sets isomorphic to  $\mathbb{C}^3$ . The transition map is

$$(z_1, z_2, z_3) \mapsto (z_1^{-1}, z_1^{-k} z_2, z_1^{2+k} z_3).$$

In this description, the torus  $T'$  acts by

$$(t_1, t_2, t_3) \cdot (z_1, z_2, z_3) = (t_1 z_1, t_2 z_2, t_3 z_3)$$

We will consider the action of the subtorus

$$T = \{(t_1, t_2, t_3) \in T' : t_1 t_2 t_3 = 1\}$$

which preserves a canonical Calabi-Yau form.[14]

Let  $L$  be the pullback of  $\mathcal{O}_{\mathbb{P}^1}(1)$  to  $X$ . We define the Hilbert polynomial of a sheaf  $\mathcal{F}$  by

$$P_{\mathcal{F}}(n) = \chi(\mathcal{F} \otimes L^{\otimes n}).$$

Then, by Riemann-Roch,

$$P_{\mathcal{F}}(n) = (\text{ch}_2(\mathcal{F}) \cdot L)n + \chi(\mathcal{F}). \quad (2)$$

Hence the reduced Hilbert polynomial of  $\mathcal{F}$  is

$$p_{\mathcal{F}}(n) = n + \frac{\chi(\mathcal{F})}{\text{ch}_2(\mathcal{F}) \cdot L}.$$

A sheaf  $\mathcal{F}$  is called (Gieseker) semistable with respect to  $L$  if for any subsheaf  $\mathcal{G}$ , we have  $p_{\mathcal{G}}(n) \leq p_{\mathcal{F}}(n)$ , which is equivalent to

$$\frac{\chi(\mathcal{G})}{\text{ch}_2(\mathcal{G}) \cdot L} \leq \frac{\chi(\mathcal{F})}{\text{ch}_2(\mathcal{F}) \cdot L}.$$

Stable sheaf is defined with the strict inequality. For details and the construction of the moduli space of semistable sheaves, we refer to [8].

We consider the moduli space of  $L$ -(semi)stable coherent sheaves of pure dimension 1 on  $X$

$$M_d(k) = \{\mathcal{F} : P_{\mathcal{F}} = dn + 1, \mathcal{F} \text{ is } L\text{-(semi)stable}\}.$$

By the condition  $\chi(\mathcal{F}) = 1$ , semistability agrees with stability. So, there exists a perfect obstruction theory on  $M_d(k)$  [17]. Since  $X$  is not compact, the virtual cycle for  $M_d(k)$  is not well-defined. However,  $T$ -action on  $X$  induces a  $T$ -action on the moduli space and the fixed

locus of this action is compact.<sup>1</sup> So, we can define an invariant by a residue integral on the fixed locus using the virtual localization formula [6]. Following [10], we call it the genus zero BPS invariant.

**Definition 2.1.** Let  $M_i^T$  be connected component of  $T$ -fixed locus  $M_d(k)^T$ . Let  $N_i^{\text{vir}}$  be the virtual normal bundle to  $M_i^T$  obtained from the moving part of the virtual tangent space. We define the *genus zero BPS invariants* by

$$n_d(k) = \sum_i \int_{[M_i^T]^{\text{vir}}} \frac{1}{e(N_i^{\text{vir}})}.$$

Here,  $e(-)$  is the equivariant Euler class.

**Lemma 2.2.** *If  $\mathcal{F} \in M_d(k)$ , then the scheme theoretic support of  $\mathcal{F}$  is a subscheme of the total space  $Y$  of  $\mathcal{O}_{\mathbb{P}^1}(k)$ .*

*Proof.* The ideal sheaf of  $Y$  is  $L^{2+k}$ . We have an exact sequence

$$\mathcal{F} \otimes L^{2+k} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}|_Y \longrightarrow 0.$$

Since  $2+k$  is a positive number, by the stability of  $\mathcal{F}$ , the first map is zero, and hence the map  $\mathcal{F} \rightarrow \mathcal{F}|_Y$  is an isomorphism.  $\square$

So, we can consider  $\mathcal{F}$  as a sheaf on  $Y$ . We will also denote by  $L$  the pullback of  $\mathcal{O}_{\mathbb{P}^1}(1)$  to  $Y$ . Then,  $M_d(k)$  is the moduli space of  $L$ -stable sheaves on  $Y$ .

Note that the zero section of  $\mathbb{P}^1$  is the only compact  $T$ -invariant curve in  $Y$ . Hence if a sheaf  $\mathcal{F}$  is  $T$ -fixed, its reduced support must be  $\mathbb{P}^1$ . In the next section, we will describe  $T$ -fixed sheaves supported on  $\mathbb{P}^1$  using toric geometry.

### 3. EQUIVARIANT SHEAVES

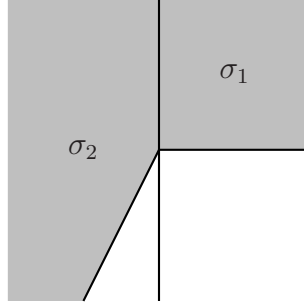
As a toric variety,  $Y$  contains a two dimensional torus  $(\mathbb{C}^*)^2$  which is isomorphic to  $T$  by the isomorphism

$$(\mathbb{C}^*)^2 \ni (t_1, t_2) \mapsto (t_1, t_2, t_1^{-1}t_2^{-1}) \in T.$$

The action of this torus is the same as the restriction of  $T$ -action on  $Y$ . So, by a slight abuse of notation, we also denote this embedded torus by  $T$  and consider  $T$ -equivariant sheaves on  $Y$ . It is well known that a stable sheaf on  $Y$  supported on a compact subscheme is  $T$ -fixed if and only if it is  $T$ -equivariant. See for example [12, 9].

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<sup>1</sup>We will see in Section 6 that we can embed  $M_d(k)$  into a compact moduli space via an embedding of  $X$  into the Hirzebruch surface. The torus fixed loci supported on  $\mathbb{P}^1$  are the same. So, they must be compact.

FIGURE 1. Toric fan of  $Y$ 

In this section, we follow [12] and describe pure equivariant sheaves  $\mathcal{F}$  on  $Y$ , and propose a new way to understand it. Let  $M$  be the group of characters of  $T$  and  $N$  be the group of one parameter subgroups. Then, the fan associated to  $Y$  (which lies in  $N \otimes \mathbb{R}$ ) is

$$\{\sigma_1 = \text{Cone}((0, 1), (1, 0)), \sigma_2 = \text{Cone}((0, 1), (-1, -k))\}$$

where  $\text{Cone}(v_1, v_2)$  denote the convex cone generated by vectors  $v_1$  and  $v_2$ . The  $T$ -invariant subvariety associated to the face  $(0, 1)$  is the zero section of  $\mathbb{P}^1$ .

We have two  $T$ -invariant affine open sets  $U_{\sigma_i} = \text{Spec}(k[S_{\sigma_i}])$ ,  $i = 1, 2$ , where  $S_{\sigma_i}$  is the semigroup defined by  $\sigma_i$

$$S_{\sigma_i} = \sigma_i^\vee \cap M.$$

For a notational convenience, we let  $M^i$  be the copy of  $M$  whose elements are expressed with respect to the semigroup generator of  $S_{\sigma_i}$ , i.e.,

$$M^1 = \{m_1(1, 0) + m_2(0, 1)\} \text{ and } M^2 = \{m_1(-1, 0) + m_2(-k, 1)\}.$$

For  $m, m' \in M^i$ , we say  $m' \geq m$  if every component of  $m' - m$  is nonnegative. Note that in the standard basis of  $M$ , this means  $m' - m$  is an element of the semigroup  $S_{\sigma_i}$ .

Then, we have a decomposition into weight spaces

$$\Gamma(U_{\sigma_i}, \mathcal{F}) = \bigoplus_{m \in M^i} \Gamma(U_{\sigma_i}, \mathcal{F})_m.$$

Denote the weight space  $\Gamma(U_{\sigma_i}, \mathcal{F})_m$  by  $F^i(m)$ ,  $m = (m_1, m_2) \in M^i$ . Since  $\mathcal{F}$  is  $\mathcal{O}_Y$ -module, each  $\Gamma(U_{\sigma_i}, \mathcal{F})$  is  $M^i$ -graded  $S_{\sigma_i}$ -module. We can reformulate the  $S_{\sigma_i}$ -module structure by the following data:  $k$ -linear maps  $\chi_{m, m'}^i: F^i(m) \rightarrow F^i(m')$  for all  $m, m' \in M^i$  with  $m' \geq m$  such that

$$\chi_{m, m}^i = 1 \text{ and } \chi_{m, m''}^i = \chi_{m', m''}^i \circ \chi_{m, m'}^i. \quad (3)$$

Moreover, in our case, where the reduced support of  $\mathcal{F}$  is  $\mathbb{P}^1$ , we have the following [12, Chapter 2].

**Proposition 3.1.** *Let  $\mathcal{F}$  be a pure equivariant sheaf on  $Y$  with support  $\mathbb{P}^1$ . Then,*

- (1) *There are integers  $A_1^1, A_1^2$  and  $A \leq B$  such that  $F^i(m_1, m_2) = 0$  unless  $A_1^i \leq m_1$  and  $A \leq m_2 \leq B$ .*
- (2) *For each  $A \leq m_2 \leq B$ , the maps  $\chi_{(m_1, m_2), (m_1+1, m_2)}^i$  are all injective and the direct limit  $\varinjlim_{m_1} F^i(m_1, m_2)$  is a finite dimensional vector space denoted by  $F^i(\infty, m_2)$ .*
- (3) *For each  $A \leq m_2 \leq B$ ,*

$$F^1(\infty, m_2) \simeq F^2(\infty, m_2)$$

*and under this identification,*

$$\chi_{(\infty, m_2), (\infty, m_2+1)}^1 = \chi_{(\infty, m_2), (\infty, m_2+1)}^2,$$

*where  $\chi_{(\infty, m_2), (\infty, m_2+1)}^i = \varinjlim_{m_1} \chi_{(m_1, m_2), (m_1, m_2+1)}^i$  for sufficiently large  $m_1$ .*

*Moreover, let  $\mathcal{C}$  be the category whose objects are  $\{F^i(m), \chi_{m, m'}^i\}$  satisfying above conditions and morphisms*

$$\phi: \{F^i(m), \chi_{m, m'}^i\} \rightarrow \{G^i(m), \lambda_{m, m'}^i\}$$

*are collections of linear maps  $\phi^i(m): F^i(m) \rightarrow G^i(m)$  satisfying*

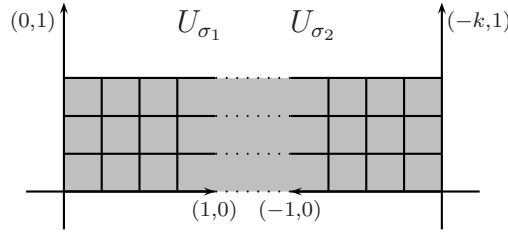
$$\phi^i(m') \circ \chi_{m, m'}^i = \lambda_{m, m'}^i \circ \phi^i(m) \text{ and } \phi^1(\infty, m_2) = \phi^2(\infty, m_2).$$

*Then, this correspondence is an equivalence between the category of pure equivariant sheaves and equivariant morphisms with the category  $\mathcal{C}$ .*

An object in the category  $\mathcal{C}$  is called  $\Delta$ -family [16]. Proposition 3.1 is a special case of more general statement about pure equivariant sheaves on a toric variety [12]. We state it for  $Y$  to avoid heavy notation.

Assume  $A$  in condition (1) is maximally chosen. Let  $\mathcal{O}_Y(\chi)$  be the structure sheaf of  $Y$  endowed with the equivariant structure induced by a character  $\chi \in M$ . Then  $\mathcal{F} \otimes \mathcal{O}_Y(\chi)$  is isomorphic to the sheaf  $\mathcal{F}$  with equivariant structure shifted by  $\chi$ . Therefore, we may assume  $A = 0$ .

**Example 3.2.** Let  $C_n$  be the  $n$ -th order thickening of  $\mathbb{P}^1$  in the direction of  $\mathcal{O}_{\mathbb{P}^1}(k)$ . More precisely,  $C_n$  is  $\text{Spec}(\text{Sym}(\mathcal{O}_{\mathbb{P}^1}(-k))/\mathfrak{I})$  where

FIGURE 2. The sheaf  $\mathcal{O}_{C_3}$ 

$\mathfrak{I}$  is the ideal generated by  $S^n(\mathcal{O}_{\mathbb{P}^1}(-k))$ . Then for the sheaf  $\mathcal{O}_{C_n}$ , we have

$$\Gamma(U_{\sigma_i}, \mathcal{O}_{C_n})_{(m_1, m_2)} = \begin{cases} \mathbb{C} & \text{if } 0 \leq m_2 \leq n-1 \text{ and } m_1 \geq 0 \\ 0 & \text{else} \end{cases}$$

We can illustrate this by putting a box at the position  $(m_1, m_2)$  if the corresponding weight space is nonzero. By the condition (3), for each open chart, the asymptotic weight vector spaces are stabilized and identified with each other. So, we place the asymptotic vector spaces in the middle. For example, the sheaf  $\mathcal{O}_{C_3}$  can be depicted as in Figure 2.

In this particular example, all weight spaces are one dimensional. We will see other examples that weight spaces have more than one dimension.

From this description, it is clear that the equivariant version of Grothendieck's theorem holds.

**Theorem 3.3.** *Let  $\mathcal{E}$  be a equivariant vector bundle of rank  $r$  on  $\mathbb{P}^1$ . Then there are integers  $a_1, \dots, a_r$  uniquely determined up to order such that we have an equivariant isomorphism  $\mathcal{E} \simeq \mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_r)$ .*

*Proof.* This theorem is due to Klyachko [11]. Since the scheme theoretic support is  $\mathbb{P}^1$ , we must have  $A = 0$  and  $B = 0$  in the condition (1) of Proposition 3.1. Let  $(\{E^1(m, 0)\}, \{E^2(m, 0)\})$  be the corresponding family. Then, we can pick a basis  $\{v_i\}$  of the asymptotic weight space  $E^1(\infty, 0) \simeq E^2(\infty, 0)$  such that for any  $m$  and  $i = 1, 2$ , a subset of  $\{v_i\}$  forms a basis of  $E^i(m, 0)$ . Therefore, by taking subfamilies generated by each  $v_i$ ,  $(\{E^1(m, 0)\}, \{E^2(m, 0)\})$  decomposes into families with one dimensional weight spaces. Hence,  $\mathcal{E}$  decomposes equivariantly into equivariant line bundles.  $\square$

Let  $U_i$  be the intersection of the open set  $U_{\sigma_i}$  with  $\mathbb{P}^1$  for  $i = 1, 2$ . Then  $\{U_i\}$  be an affine open cover of  $\mathbb{P}^1$ . We fix an  $T$ -equivariant

structure of  $\mathcal{O}_{\mathbb{P}^1}(k)$  by the weight space decomposition on each open set

$$\Gamma(U_1, \mathcal{O}_{\mathbb{P}^1}(k)) = \bigoplus_{m \geq 0} \mathbb{C}_m, \quad \Gamma(U_2, \mathcal{O}_{\mathbb{P}^1}(k)) = \bigoplus_{m \leq k} \mathbb{C}_m,$$

where  $\mathbb{C}_m$  is a one dimensional representation of  $T$  with character  $\chi(t_1, t_2) = t_1^m$ ,  $m \in \mathbb{Z}$ . Then, given an equivariant sheaf  $\mathcal{F}$  on  $\mathbb{P}^1$  with

$$\Gamma(U_i, \mathcal{F})_m = F^i(m),$$

we have a natural equivariant structure on  $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(k)$  by

$$\begin{aligned} \Gamma(U_1, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(k))_m &= F^1(m) \\ \Gamma(U_2, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(k))_m &= F^2(m - k). \end{aligned} \tag{4}$$

Now, we consider  $j$ -th row of the weight space decompositions. Given an equivariant sheaf  $\mathcal{F}$  on  $Y$ , let  $\mathcal{F}_j$  be the sheaf defined by

$$\Gamma(U_i, \mathcal{F}_j)_{(m_1, m_2)} = \begin{cases} \Gamma(U_i, \mathcal{F})_{(m_1, m_2)} & \text{if } m_2 = j \\ 0 & \text{else} \end{cases}$$

Then,  $\mathcal{F}_j$  has scheme theoretic support  $\mathbb{P}^1$  and hence decomposes into equivariant line bundles by Theorem 3.3.

**Theorem 3.4.** *Let  $\mathcal{F}$  be a pure  $T$ -equivariant sheaf on  $Y$ . Then  $\mathcal{F}$  is determined by the following data: For  $0 \leq j \leq B$ ,  $\mathcal{F}_j \simeq \bigoplus_{i=1}^{d_j} \mathcal{O}_{\mathbb{P}^1}(a_{ij})$  and equivariant morphisms*

$$\phi_j: \mathcal{F}_j \rightarrow \mathcal{F}_{j+1} \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

such that for all  $t_1 = (t_1, 1) \in T$ , there exist isomorphisms  $\alpha_j: t_1^* \mathcal{F}_j \rightarrow \mathcal{F}_j$  such that the diagram

$$\begin{array}{ccc} t_1^* \mathcal{F}_j & \xrightarrow{t_1^* \phi_j} & t_1^* (\mathcal{F}_{j+1} \otimes \mathcal{O}_{\mathbb{P}^1}(k)) \\ \alpha_j \downarrow & & \downarrow \alpha_{j+1} \otimes \mu \\ \mathcal{F}_j & \xrightarrow{\phi_j} & \mathcal{F}_{j+1} \otimes \mathcal{O}_{\mathbb{P}^1}(k) \end{array} \tag{5}$$

commutes, where  $\mu: t_1^*(\mathcal{O}_{\mathbb{P}^1}(k)) \simeq \mathcal{O}_{\mathbb{P}^1}(k)$  is given by the equivariant structure of  $\mathcal{O}_{\mathbb{P}^1}(k)$  fixed in the above discussion.

*Proof.* The horizontal maps  $\chi_{(m_1, j), (m_1+1, j)}^i$  will determine the decomposition  $\mathcal{F}_j \simeq \bigoplus_{i=1}^{d_j} \mathcal{O}_{\mathbb{P}^1}(a_{ij})$  by Theorem 3.3. It remains to consider the vertical maps  $\chi_{(m_1, j), (m_1, j+1)}^i$ .



Recall that we are using different basis of  $M$  for  $\chi^1$  and  $\chi^2$ . The  $(m_1, j)$  in the subscript means  $m_1(1, 0) + j(0, 1)$  for  $\chi^1$  and  $m_1(-1, 0) + j(-k, 1)$  for  $\chi^2$ . Rewrite in the standard basis of  $M$ ,

$$\begin{aligned}\chi_{(m_1, j), (m_1, j+1)}^1 &: F^1(m_1, j) \rightarrow F^1(m_1, j+1) \\ \chi_{(m_1, j), (m_1, j+1)}^2 &: F^2(-m_1 - kj, j) \rightarrow F^2(-m_1 - kj - k, j+1).\end{aligned}$$

Thus, this will define an equivariant morphism

$$\phi_j: \mathcal{F}_j \rightarrow \mathcal{F}_{j+1} \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

by (4).

Since  $\mathcal{F}$  is equivariant,  $t_1^* \mathcal{F} \simeq \mathcal{F}$ . Hence, there exist isomorphisms  $\alpha_j$ 's such that the diagram above commutes by Remark 3.5 and Theorem 3.6.

Conversely, if we have such isomorphisms  $\alpha_j: t_1^* \mathcal{F}_j \simeq \mathcal{F}_j$ , each  $\mathcal{F}_j$  is equivariant [9], so we have an weight space decomposition and horizontal maps  $\chi_{(m_1, j), (m_1+1, j)}^i$ . The equivariant morphisms  $\phi_j$  define  $\chi_{(m_1, j), (m_1, j+1)}^i$ . By the commutativity of (5), they commute with each other. Hence the data  $(\mathcal{F}_j, \phi_j, \alpha_j)$  determines  $\mathcal{F}$ , by Proposition 3.1.  $\square$

*Remark 3.5.* Let  $\pi: Y \rightarrow \mathbb{P}^1$  be the natural projection. In the above theorem, it is clear that

$$\pi_* \mathcal{F} \simeq \bigoplus_{j=0}^B \bigoplus_{i=1}^{d_j} \mathcal{O}_{\mathbb{P}^1}(a_{ij}).$$

$\pi_*$  induces an equivalence between the category of  $\mathcal{O}_Y$ -modules and the category of  $\pi_* \mathcal{O}_Y$ -modules on  $\mathbb{P}^1$  [7, Ex.II.5.17]. Since  $\pi_* \mathcal{O}_Y \simeq \mathcal{O}_{\mathbb{P}^1}(-k)$ ,  $\pi_* \mathcal{O}_Y$ -modules structure on  $\pi_* \mathcal{F}$  is given by a map

$$\pi_* \mathcal{F} \rightarrow \pi_* \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

The previous theorem shows that if  $\mathcal{F}$  is a pure equivariant sheaf, this map is given by  $\phi_j$ 's. In this sense, we will call the collection  $\{\mathcal{F}_j, \phi_j\}$  associated to a sheaf  $\mathcal{F}$  a  $\pi_* \mathcal{O}_Y$ -modules structure of  $\mathcal{F}$ .

**Theorem 3.6.** *Let  $\mathcal{F}$  and  $\mathcal{G}$  be  $T$ -equivariant sheaves on  $Y$  whose  $\pi_* \mathcal{O}_Y$ -modules structure are  $\{\mathcal{F}_j, \phi_j\}$  and  $\{\mathcal{G}_j, \psi_j\}$  respectively. Then  $\mathcal{F}$  and  $\mathcal{G}$  are isomorphic to each other if and only if there exist isomorphisms  $\mu_j: \mathcal{F}_j \rightarrow \mathcal{G}_j$  such that  $\mu_{j+1} \circ \phi_j = \psi_{j+1} \circ \mu_j$ .*

*Proof.* This is a straightforward consequence of the equivalence between the category of  $\mathcal{O}_Y$ -modules and the category of  $\pi_* \mathcal{O}_Y$ -modules on  $\mathbb{P}^1$ .  $\square$

**Theorem 3.7.** *Let  $P_{\mathcal{F}}(n) = \chi(\mathcal{F} \otimes L^{\otimes n}) = dn + \chi(\mathcal{F})$ . Then*

$$d = \sum_{j=0}^B d_j \text{ and } \chi(\mathcal{F}) = \sum_{j=0}^B \sum_{i=1}^{d_j} (a_{ij} + 1)$$

*Proof.* The first equation is clear since the support of  $\mathcal{F}$  has multiplicity  $\sum_{j=0}^B d_j$  along  $\mathbb{P}^1$ . The second equation follows from  $\chi(\mathcal{F}) = \chi(\pi_* \mathcal{F})$  since  $\pi$  is affine.  $\square$

To test the stability, we only need to test for equivariant subsheaves.

**Proposition 3.8.** *Suppose  $X$  is a projective variety with a torus action. Let  $\mathcal{F}$  be a pure equivariant sheaf on  $X$ . Then  $\mathcal{F}$  is (Gieseker) stable if and only if  $p_{\mathcal{G}} < p_{\mathcal{F}}$  for any proper equivariant subsheaf  $\mathcal{G}$ .*

*Proof.* [12, Proposition 3.19]  $\square$

Therefore, a sheaf  $\mathcal{F}$  associated to  $\{\mathcal{F}_j, \phi_j\}$  is stable if and only if for any  $\pi_* \mathcal{O}_Y$ -submodule  $\mathcal{G} = \{\mathcal{G}_j, \psi_j\}$ , i.e., a collection of equivariant subsheaves  $\mathcal{G}_j \subset \mathcal{F}_j$  compatible with  $\phi_i$ , we have

$$\frac{\chi(\mathcal{G})}{r(\mathcal{G})} < \frac{\chi(\mathcal{F})}{d}.$$

where  $r(\mathcal{G})$  is the multiplicity of  $\mathcal{G}$  along  $\mathbb{P}^1$ .

**Definition 3.9.** For a pure equivariant sheaf  $\mathcal{F}$  as in Theorem 3.4, we will call  $(d_0, d_1, \dots, d_B)$  the *type* of  $\mathcal{F}$ .

#### 4. ENUMERATION OF EQUIVARIANT SHEAVES

Using the classification given in the previous section, we want to count the (virtual) number of  $T$ -equivariant sheaves.

**Definition 4.1.** Let  $M_{(d_0, \dots, d_B)}^T(k)$  denote the subscheme of  $M_d(k)$  which consists of stable  $T$ -equivariant sheaves of type  $(d_0, \dots, d_B)$  with  $d = \sum_{j=0}^B d_j$ . We define

$$N_d(k) = e_{\text{top}}(M_d(k)),$$

$$N_{(d_0, \dots, d_B)}(k) = e_{\text{top}}(M_{(d_0, \dots, d_B)}^T(k))$$

where  $e_{\text{top}}(-)$  is the topological Euler characteristic.

It is clear from the localization formula that

$$N_d(k) = \sum_{(d_0, \dots, d_B) \vdash d} N_{(d_0, \dots, d_B)}(k), \quad (6)$$

where the sum runs over the set of all ordered partitions of  $d$ .

In Section 6, we will show

$$n_d(k) = (-1)^{kd^2+1} e_{\text{top}}(M_d(k)). \quad (7)$$

Hence it is enough to compute  $N_d(k)$ .

**4.1. Type  $(1^d)$ .** Let  $(1^d)$  denote  $(1, 1, \dots, 1)$  with 1 repeated  $d$  times. Let  $\mathcal{F}$  be a  $T$ -equivariant sheaf of type  $(1^d)$  whose  $\pi_* \mathcal{O}_Y$ -module structure is  $\{\mathcal{F}_j, \phi_j\}$ . Assume  $\mathcal{F}_j \simeq \mathcal{O}_{\mathbb{P}^1}(a_j)$  for  $0 \leq j \leq d-1$ . Then, since  $\chi(\mathcal{F}) = 1$ , we have

$$\sum_{j=0}^{d-1} (a_j + 1) = 1.$$

Let  $x$  and  $y$  be homogeneous coordinates of  $\mathbb{P}^1$ . By the condition (5) in Theorem 3.4, the map  $\phi_j$  is given by a monomial in  $x$  and  $y$  of degree  $a_{j+1} - a_j + k$ .

**Proposition 4.2.**  $\mathcal{F}$  of type  $(1^d)$  is stable if and only if  $\phi_j$ 's are all nonzero and

$$\sum_{j=0}^{h-1} (a_j + 1) \geq 1$$

for any  $1 \leq h \leq d$ .

*Proof.* Since  $\mathcal{F}$  is indecomposable,  $\phi_j$ 's are all nonzero. To check the stability, it is enough to check for the subsheaf  $\mathcal{G}$  with

$$\mathcal{G}_j = \begin{cases} \mathcal{F}_j & \text{if } j \geq h \\ 0 & \text{else} \end{cases}$$

for  $0 \leq h \leq d-1$ . Hence, the stability condition is

$$\sum_{j=0}^{h-1} (a_j + 1) \geq 1$$

where the left side is the Euler characteristic of  $\mathcal{F}/\mathcal{G}$ .  $\square$

**Corollary 4.3.**  $N_{(1^d)}$  is equal to

$$\sum_{\lambda_{d-1} \geq \dots \geq \lambda_0 \geq 0} \prod_{j=0}^{d-2} (\lambda_{j+1} - \lambda_j + 1)$$

where the sum runs over all  $\lambda_{d-1} \geq \dots \geq \lambda_0 \geq 0$  such that

$$\sum_{j=0}^{d-1} \lambda_j = \frac{d(d-1)}{2} k - (d-1)$$

and for any  $1 \leq h \leq d$ ,

$$\sum_{j=0}^{h-1} \lambda_j \geq \frac{h(h-1)}{2}k - (h-1).$$

*Proof.* Since  $\phi_j$  is nonzero, we have  $a_j \leq a_{j+1} + k$ . We let

$$\lambda_j = a_j + jk$$

so that  $\lambda_{d-1} \geq \lambda_{d-2} \geq \cdots \geq \lambda_0 \geq 0$ . Then,  $\phi_j$  is a monomial of degree

$$a_{j+1} - a_j + k = \lambda_{j+1} - \lambda_j.$$

By Theorem 3.6, each coefficient of the monomial  $\phi_j$  can be set to be 1 by scaling isomorphisms. So, we have  $\lambda_{j+1} - \lambda_j + 1$  choices for  $\phi_j$ . The condition for  $\lambda_j$ 's can be easily seen to be equivalent to the condition in  $a_j$ 's in the previous proposition.  $\square$

**4.2. Types  $(n, 1^d)$  and  $(1^d, n)$ .** We will use the following lemma frequently.

**Definition 4.4.** For a monomial  $\alpha$  in  $x$  and  $y$ , we set  $\gcd(\alpha, 0) = \alpha$ . Hence,  $\deg(\gcd(\alpha, 0)) = \deg(\alpha)$ .

**Lemma 4.5.** *Suppose*

$$\phi = (\alpha_1, \alpha_2): \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(b) \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

$$\psi = (\beta_1, \beta_2)^t: \mathcal{O}_{\mathbb{P}^1}(c) \rightarrow (\mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)) \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

are nonzero maps between sheaves on  $\mathbb{P}^1$  where  $\alpha_1, \alpha_2, \beta_1, \beta_2$  are monomials of appropriate degrees in the homogeneous coordinates  $x$  and  $y$ . Let  $K$  be the kernel of  $\phi$  and  $Q$  be such that  $Q \otimes \mathcal{O}_{\mathbb{P}^1}(k)$  be the torsion free part of the cokernel of  $\psi$ . Then

$$\deg K = a_1 + a_2 - b - k + \deg(\gcd(\alpha_1, \alpha_2)), \quad (8)$$

$$\deg Q = d_1 + d_2 - c + k - \deg(\gcd(\beta_1, \beta_2)). \quad (9)$$

*Proof.* Let  $r = \deg(\gcd(\alpha_1, \alpha_2))$ . If either of  $\alpha_1$  or  $\alpha_2$  is zero, by symmetry, we may assume  $\alpha_1$  is zero. Then,  $\alpha_2$  is nonzero monomial of degree  $b - a_2 + k$ . So,  $K \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)$  and we have (8). Now, suppose  $\alpha_1$  and  $\alpha_2$  are both nonzero. Since  $\alpha_i$  is a monomial of degree  $b - a_i + k$ , there are monomials  $p$  and  $q$  of degree  $b - a_1 + k - r$  and  $b - a_2 + k - r$  respectively, such that

$$q\alpha_1 = p\alpha_2.$$

Then the image of the inclusion

$$\mathcal{O}_{\mathbb{P}^1}(a_1 + a_2 - b - k + r) \xrightarrow{\begin{pmatrix} q \\ -p \end{pmatrix}} \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$$

is  $K$ . Therefore, we have (8).

The proof of (9) is similar.  $\square$

**Definition 4.6.** Given a map  $\psi$  as in the above lemma, we will call  $Q$  the *torsion free cokernel* of  $\psi$ .

We start with types  $(n, 1)$  and  $(1, n)$ . Let  $\mathcal{F}$  be a  $T$ -equivariant sheaf of type  $(n, 1)$  which corresponds to the collection  $(\{\mathcal{F}_0, \mathcal{F}_1\}, \phi)$ . Assume  $\mathcal{F}_0 \simeq \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$  and  $\mathcal{F}_1 \simeq \mathcal{O}_{\mathbb{P}^1}(b)$  and  $\phi = (\alpha_1, \dots, \alpha_n)$ , where

$$\alpha_i: \mathcal{O}_{\mathbb{P}^1}(a_i) \rightarrow \mathcal{O}_{\mathbb{P}^1}(b) \otimes \mathcal{O}_{\mathbb{P}^1}(k).$$

Then,  $\chi(\mathcal{F}) = 1$  is equivalent to

$$\sum_{i=1}^n (a_i + 1) + (b + 1) = 1. \quad (10)$$

As before, by condition (5) in Theorem 3.4,  $\alpha_i$  is given by a monomial in  $x$  and  $y$  of degree  $b - a_i + k$ .

**Proposition 4.7.**  $\mathcal{F}$  of type  $(n, 1)$  is stable if and only if

- $\alpha_i$ 's are nonzero,
- $a_i \geq 0$ ,
- for all  $1 \leq i, j \leq n$ ,  $\deg(\gcd(\alpha_i, \alpha_j)) \leq b - a_i - a_j + k - 1$ .

*Proof.* As before, for  $\mathcal{F}$  to be indecomposable,  $\alpha_i$ 's are nonzero. Let  $\mathcal{G}$  be a subsheaf of  $\mathcal{F}$  whose  $\pi_* \mathcal{O}_Y$ -module structure is  $(\{\mathcal{G}_0, \mathcal{G}_1\}, \psi)$ . Since  $\mathcal{F}_1$  is of rank 1, we have two cases:  $\mathcal{G}_1 = \mathcal{F}_1$  or  $\mathcal{G}_1 = 0$ .

Suppose  $\mathcal{G}_1 = \mathcal{F}_1$ . Let  $\mathcal{G}_0 \simeq \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a'_i)$  where  $r$  is the rank of  $\mathcal{G}_0$ . Without loss of generality, we may assume  $a_i$ 's and  $a'_i$ 's are nonincreasing. Then for  $1 \leq i \leq r$ ,

$$a'_i \leq a_i$$

because otherwise there does not exist an injective map from  $\mathcal{G}_0$  to  $\mathcal{F}_0$ . So, it is enough to check for the cases  $a'_i = a_i$ , i.e.,  $\mathcal{G}_0 \simeq \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$  for some  $0 \leq r \leq n$ . Then by looking at the quotients, it is easy to see that the stability implies  $a_i \geq 0$  for all  $1 \leq i \leq n$ . Note that if  $\mathcal{G}_0 = 0$ , we have  $b \leq -1$  which is a consequence of (10) and  $a_i \geq 0$ .

Now suppose  $\mathcal{G}_1 = 0$ . Then,  $\mathcal{G}_0$  is a subsheaf of  $K = \ker \phi$ . Let  $K_{ij}$  be the kernel of the restricted map

$$(\alpha_i, \alpha_j): \mathcal{O}_{\mathbb{P}^1}(a_i) \oplus \mathcal{O}_{\mathbb{P}^1}(a_j) \rightarrow \mathcal{O}_{\mathbb{P}^1}(b) \otimes \mathcal{O}_{\mathbb{P}^1}(k).$$

When  $\mathcal{G}_0 = K_{ij}$ , by (8), the stability implies

$$a_i + a_j - b - k + \deg(\gcd(\alpha_i, \alpha_j)) \leq -1,$$

which is the third condition. For an arbitrary  $\mathcal{G}_0$ , it suffices to show the degree of  $\mathcal{G}_0$  is negative provided that the degrees of  $K_{ij}$  are negative for all  $1 \leq i, j \leq n$ . Hence we may assume that  $\mathcal{G}_0$  is a line bundle. By Proposition 3.8, we may also assume  $\mathcal{G}_0$  is equivariant subsheaf of  $\mathcal{F}_0$ , that is, the inclusion

$$\mathcal{G}_0 \rightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i) \simeq \mathcal{F}_0$$

is given by a matrix with monomial entries. Let  $(p_1, \dots, p_n)^t$  be the inclusion map where  $p_i$ 's are monomials. Then, we have

$$\sum_{i=1}^n p_i \alpha_i = 0.$$

Since all terms are monomials and at least two terms are nonzero, this implies that there exist  $j_1, j_2$  such that  $p_{j_1} \alpha_{j_1}$  and  $p_{j_2} \alpha_{j_2}$  are nonzero and proportional. Then  $\deg(p_{j_1} \alpha_{j_1}) \geq \deg(\text{lcm}(\alpha_{j_1}, \alpha_{j_2}))$ , and

$$\begin{aligned} \deg \mathcal{G}_0 &= a_{j_1} - \deg(p_{j_1}) \leq a_{j_1} + \deg(\alpha_{j_1}) - \deg(\text{lcm}(\alpha_{j_1}, \alpha_{j_2})) \\ &= a_{j_1} - \deg(\alpha_{j_2}) + \deg(\gcd(\alpha_{j_1}, \alpha_{j_2})) \\ &= a_{j_1} + a_{j_2} - b - k + \deg(\gcd(\alpha_{j_1}, \alpha_{j_2})) \\ &= \deg K_{j_1, j_2} \leq -1. \end{aligned}$$

Hence it is enough to check for subsheaves  $K_{ij}$ . □

The type  $(1, n)$  is dual to the type  $(n, 1)$ . Now, assume  $\mathcal{F}_0 \simeq \mathcal{O}_{\mathbb{P}^1}(c)$  and  $\mathcal{F}_1 \simeq \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(d_i)$  and  $\phi = (\beta_1, \dots, \beta_n)^t$ , where

$$\beta_i: \mathcal{O}_{\mathbb{P}^1}(c) \rightarrow \mathcal{O}_{\mathbb{P}^1}(d_i) \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

is given by a monomial in  $x$  and  $y$  of degree  $d_i - c + k$ . Then,  $\chi(\mathcal{F}) = 1$  is equivalent to

$$(c+1) + \sum_{i=1}^n (d_i+1) = 1. \quad (11)$$

**Proposition 4.8.**  $\mathcal{F}$  of type  $(1, n)$  is stable if and only if

- $\beta_i$ 's are nonzero,
- $d_i \leq -1$ ,
- for all  $1 \leq i, j \leq n$ ,  $\deg(\gcd(\beta_i, \beta_j)) \leq d_i + d_j - c + k$ .

*Proof.* The proof is dual to the proof of the previous proposition. Indecomposability implies  $\beta_i$ 's are nonzero. Let  $\mathcal{G}$  be a subsheaf of  $\mathcal{F}$  whose  $\pi_* \mathcal{O}_Y$ -module structure is  $(\{\mathcal{G}_0, \mathcal{G}_1\}, \psi)$ . If  $\mathcal{G}_0 = 0$ , it is enough to check for  $\mathcal{G}_1 \simeq \mathcal{O}_{\mathbb{P}^1}(d_i)$ . So, we have  $d_i \leq -1$ .

Suppose  $\mathcal{G}_0 = \mathcal{O}_{\mathbb{P}^1}(c)$ . Let  $Q_{ij}$  be the torsion free cokernel of the map

$$\beta_{ij} = (\beta_i, \beta_j)^t: \mathcal{O}_{\mathbb{P}^1}(c) \rightarrow (\mathcal{O}_{\mathbb{P}^1}(d_i) \oplus \mathcal{O}_{\mathbb{P}^1}(d_j)) \otimes \mathcal{O}_{\mathbb{P}^1}(k).$$

If  $\mathcal{G}_1$  is the saturation of  $\bigoplus_{t \neq i,j} \mathcal{O}_{\mathbb{P}^1}(d_t) \oplus \text{im} \beta_{ij}$ , then by (9),

$$\deg Q_{ij} = d_i + d_j - c + k - \deg(\gcd(\beta_i, \beta_j)) \geq 0,$$

which is the third condition. Now, let  $\mathcal{G}_1$  be an arbitrary subsheaf of  $\mathcal{F}_1$  containing the image of  $\phi$ . We may assume  $\mathcal{G}_1$  is an equivariant saturated subsheaf of rank  $n - 1$ . Let  $(q_1, \dots, q_n)$  be the natural projection map from  $\mathcal{F}_1$  to the quotient  $\mathcal{F}_1/\mathcal{G}_1$  where  $q_i$ 's are monomials. Then

$$\sum_{i=1}^n \beta_i q_i = 0.$$

As in the previous proposition, we can find  $j_1, j_2$  such that  $\beta_{j_1} q_{j_1}$  and  $\beta_{j_2} q_{j_2}$  are nonzero and proportional. Thus,

$$\begin{aligned} \deg \mathcal{F}_1/\mathcal{G}_1 &= d_{j_1} + \deg(q_{j_1}) \geq d_{j_1} + \deg(\beta_{j_2}) - \deg(\gcd(\beta_{j_1}, \beta_{j_2})) \\ &= d_{j_1} + d_{j_2} - c + k - \deg(\gcd(\beta_{j_1}, \beta_{j_2})) \\ &= \deg Q_{j_1, j_2} \geq 0. \end{aligned}$$

So, it is enough to check for  $Q_{ij}$ . □

Propositions 4.7 and 4.8 have straightforward generalizations to types  $(n, 1^d)$  and  $(1^d, n)$ .

**Proposition 4.9.** *For a sheaf  $\mathcal{F}$  of type  $(n, 1^d)$ , let  $\mathcal{F}_0 \simeq \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$  and  $\mathcal{F}_j \simeq \mathcal{O}_{\mathbb{P}^1}(b_j)$  for  $1 \leq j \leq d$ . By  $\chi(\mathcal{F}) = 1$ , we have*

$$\sum_{i=1}^n (a_i + 1) + \sum_{j=1}^d (b_j + 1) = 1. \quad (12)$$

Then,  $\mathcal{F}$  of type  $(n, 1^d)$  is stable if and only if

- all maps  $\phi_j$ ,  $0 \leq j \leq d$  have nonzero monomial entries,
- $a_i \geq 0$ ,  $\sum_{j=s}^d (b_j + 1) \leq 0$  for  $1 \leq s \leq d$ ,
- $\deg(\gcd(\alpha_i, \alpha_j)) \leq b_1 - a_i - a_j + k - 1$ ,

where  $\phi_0 = (\alpha_1, \dots, \alpha_n)$ .

**Proposition 4.10.** *For a sheaf  $\mathcal{F}$  of type  $(1^d, n)$ , let  $\mathcal{F}_j \simeq \mathcal{O}_{\mathbb{P}^1}(c_j)$  for  $0 \leq j \leq d - 1$  and  $\mathcal{F}_d \simeq \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(d_i)$ . By  $\chi(\mathcal{F}) = 1$ , we have*

$$\sum_{j=0}^{d-1} (c_j + 1) + \sum_{i=1}^n (d_i + 1) = 1. \quad (13)$$

Then,  $\mathcal{F}$  of type  $(1^d, n)$  is stable if and only if

- all maps  $\phi_j$ ,  $0 \leq j \leq d$  have nonzero monomial entries,
- $d_i \leq -1$ ,  $\sum_{j=0}^s (d_j + 1) \geq 1$  for  $0 \leq s \leq d-1$ ,
- $\deg(\gcd(\beta_i, \beta_j)) \leq d_i + d_j - c_{d-1} + k$ ,

where  $\phi_{d-1} = (\beta_1, \dots, \beta_n)^t$ .

**Corollary 4.11.** *All stable equivariant sheaves of type  $(1^d)$ ,  $(n, 1^d)$  or  $(1^d, n)$  are isolated points in  $M_d(k)^T$ .*

*Proof.* By scaling automorphisms in each case, we can set the coefficients of monomials to be 1. So, equivariant sheaves of these types are isolated.  $\square$

**Corollary 4.12.** *For any  $k \geq -1$ ,*

$$N_{(1,n)}(k) = N_{(n,1)}(k+n-1)$$

*Proof.* For a given  $c$  and  $d_j$ 's as in Proposition 4.8, we let  $a_j = -1 - d_j$  and  $b = -n - c$  and  $\alpha_j = \beta_j$ . Note that  $\deg(\beta_j) = d_j - c + k = c - a_j + (k + n - 1)$  as required. Moreover,  $b - a_i - a_j + ((k + n - 1) - 1) = d_i + d_j - c + k$ , and the equation (10) for the Euler characteristic is equivalent to (11). So,  $a_j$ ,  $b$ ,  $\alpha_j$  so defined will determine a stable sheaf in  $M_{(n,1)}^T(k+n-1)$ . Hence, this gives a bijection between  $M_{(1,n)}^T(k)$  and  $M_{(n,1)}^T(k+n-1)$ .  $\square$

Now, we can actually count stable equivariant sheaves when  $d = 1, 2$ , or  $3$ , because only sheaves of above types appear. Conjecture 1.1 combined with the Gromov-Witten theory [3] predicts that

$$n_1(k) = (-1)^{k+1}, \quad (14)$$

$$n_2(k) = \begin{cases} -\frac{k(k+2)}{4} & \text{if } k \text{ is even,} \\ -\frac{(k+1)^2}{4} & \text{if } k \text{ is odd,} \end{cases} \quad (15)$$

$$n_3(k) = (-1)^{k+1} \frac{k(k+1)^2(k+2)}{6}. \quad (16)$$

By (7), signs are correct. By Corollary 4.11 and the localization formula (6), we compute  $N_d(k) = e_{\text{top}}(M_d(k))$  by counting  $T$ -equivariant sheaves.

4.3.  $d = 1$ . By Corollary 4.3, it is easy to see that  $N_1(k) = 1$ . We can see this more directly. Let  $\mathcal{F}$  be a stable sheaf with Hilbert polynomial  $n+1$  whose support is  $\mathbb{P}^1$ . Then  $\mathcal{F}$  has a section, or a nonzero morphism  $\mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{F}$ . Since  $\mathcal{O}_{\mathbb{P}^1}$  is stable with the Hilbert polynomial  $n+1$ , this morphism is an isomorphism. Hence

$$M_1^T(k) = \{\mathcal{O}_{\mathbb{P}^1}\}.$$



Hence, we have

$$N_1(k) = 1.$$

4.4.  $d = 2$ . Only sheaves of type  $(1, 1)$  appear. By Corollary 4.3,

$$N_2(k) = \sum_{\lambda_1 \geq \lambda_0} (\lambda_1 - \lambda_0 + 1)$$

where the sum is over all partitions  $\lambda_1 + \lambda_0 = k - 1$ . Therefore,

$$N_2(k) = \sum_{\lambda_0=0}^{\lfloor \frac{k-1}{2} \rfloor} (k - 2\lambda_0) = \begin{cases} \frac{k(k+2)}{4} & \text{if } k \text{ is even.} \\ \frac{(k+1)^2}{4} & \text{if } k \text{ is odd.} \end{cases}$$

4.5.  $d = 3$ . In this case, sheaves of type  $(1, 1, 1)$ ,  $(2, 1)$  and  $(1, 2)$  appear. By Corollary 4.12,

$$N_{(1,2)}(k) = N_{(2,1)}(k+1). \quad (17)$$

We start with the type  $(2, 1)$ .

To count the  $T$ -equivariant sheaves of type  $(2, 1)$ , we let

$$S_{(2,1)}(k) = \left\{ (a_1, a_2, b) \in \mathbb{Z}^3 : \begin{array}{l} a_1 + a_2 + b = -2, \\ 0 \leq a_1, a_2 \leq b + k, \quad b \leq -1 \end{array} \right\}.$$

For  $(a_1, a_2, b) \in S_{(2,1)}(k)$ , we count pairs  $(\alpha_1, \alpha_2)$  of monomials which do not have a common factor of degree greater than  $b - a_1 - a_2 + k - 1 = 2b + k + 1$ . But if  $a_1 = a_2 = a$ , switching two factors of  $\mathcal{F}_0 = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$  gives an isomorphism between two sheaves determined by  $(\alpha_1, \alpha_2)$  and  $(\alpha_2, \alpha_1)$ . So, we must count half of such pairs  $(\alpha_1, \alpha_2)$  if the degree of  $\alpha_1$  and  $\alpha_2$  are the same.

**Definition 4.13.** For  $r < \min(n, m)$ ,

$$P_{(n,m,r)} = \left\{ (v, w) : \begin{array}{l} v, w \text{ monomials in } x \text{ and } y, \\ \deg v = n, \deg w = m, \deg(\gcd(v, w)) \leq r \end{array} \right\}$$

$$f(n, m, r) = \begin{cases} |P_{(n,m,r)}| & \text{if } n \neq m \\ \frac{1}{2}|P_{(n,m,r)}| & \text{if } n = m \end{cases}$$

Then, the total number of  $T$ -fixed sheaves of type  $(2, 1)$  is

$$N_{(2,1)}(k) = \sum_{(a_1, a_2, b) \in S_{(2,1)}(k)} f(b - a_1 + k, b - a_2 + k, 2b + k + 1). \quad (18)$$

**Lemma 4.14.**  $|P_{(n,m,r)}| = \begin{cases} (r+1)(r+2) & \text{if } 0 \leq r < \min(n, m) \\ 0 & \text{if } r < 0 \end{cases}$

*Proof.* For  $(v, w) \in P_{(n,m,r)}$ , let  $g$  be  $\gcd(v, w)$  and  $d$  be its degree. Then  $(v, w)$  is either  $(x^{n-d}g, y^{m-d}g)$  or  $(y^{n-d}g, x^{m-d}g)$ . Since there are  $d+1$  choices for  $g$ ,  $|P_{(n,m,r)}|$  is  $2 \sum_{d=0}^r (d+1) = (r+1)(r+2)$ .  $\square$

**Lemma 4.15.** *If  $k \geq 1$ ,*

$$\begin{aligned} N_{(2,1)}(k) = & \sum_{b=\lceil -\frac{k+1}{2} \rceil}^{-1} \lfloor -\frac{b+1}{2} \rfloor (k+2b+2)(k+2b+3) \\ & + \frac{1}{2} \sum_{a=0}^{\lfloor \frac{k-3}{4} \rfloor} (k-4a-2)(k-4a-1) \end{aligned} \quad (19)$$

*Proof.* Each sum corresponds to the case  $a_1 > a_2$  and  $a_1 = a_2$  respectively. Note that in (18),  $f$  has  $\frac{1}{2}$  factor if and only if  $a_1 = a_2$ .

First, We count the case  $a_1 > a_2$ . From equation (18), since  $r = 2b + k + 1 \geq 0$ ,  $-\frac{k+1}{2} \leq b \leq -1$ . For each  $b$ , we can check there are  $\lfloor -\frac{b+1}{2} \rfloor$  pairs of  $(a_1, a_2)$  with  $a_1 > a_2$  satisfying all the required conditions. By the Lemma 4.14, this verifies the first sum.

If  $a_1 = a_2 = a$ , then  $b = -2 - 2a \geq -\frac{k+1}{2}$ . So,  $0 \leq a \leq \frac{k-3}{4}$ . Thus by (18) and Lemma 4.14, we get the second sum.  $\square$

To count sheaves of type  $(1, 1, 1)$ , let

$$S_{(1,1,1)}(k) = \left\{ (\lambda_0, \lambda_1, \lambda_2) : \sum_{i=1}^3 \lambda_i = 3k - 2, 0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq 2k - 1 \right\}$$

Then by Corollary 4.3,

$$N_{(1,1,1)}(k) = \sum_{(\lambda_0, \lambda_1, \lambda_2) \in S_{(1,1,1)}(k)} (\lambda_2 - \lambda_1 + 1)(\lambda_1 - \lambda_0 + 1). \quad (20)$$

**Theorem 4.16.**

$$N_3(k) = \frac{k(k+1)^2(k+2)}{6} \quad (21)$$

*Proof.* We compute  $N_3(k) - N_3(k-1)$  and prove (21) by induction. It remains to count type  $(1,1,1)$  sheaves.

The map

$$(\lambda_0, \lambda_1, \lambda_2) \mapsto (\lambda_0 + 1, \lambda_1 + 1, \lambda_2 + 1)$$

gives an injection from  $S_{(1,1,1)}(k-1)$  to  $S_{(1,1,1)}(k)$ . Since the summand in (20) does not change under this map, the corresponding terms are canceled in  $N_3(k) - N_3(k-1)$ .

The remaining terms in  $N_3(k)$  are for  $\lambda_0 = 0$  or  $\lambda_2 = 2k - 1$ . We claim that

$$\begin{aligned} & N_{(1,1,1)}(k) - N_{(1,1,1)}(k-1) \\ &= \sum_{\lambda_1=k-1}^{\lfloor \frac{3k-2}{2} \rfloor} (3k - 2\lambda_1 - 1)(\lambda_1 + 1) + \sum_{\lambda_0=1}^{\lfloor \frac{k-1}{2} \rfloor} (\lambda_0 + k + 1)(k - 2\lambda_0). \end{aligned}$$

If  $\lambda_0 = 0$ , then we must have  $\lambda_1 + \lambda_2 = 3k - 2$ ,  $\lambda_2 \leq 2k - 1$ . So,  $\lambda_2 = 3k - 2 - \lambda_1$  and  $k - 1 \leq \lambda_1 \leq \frac{3k-2}{2}$ . Hence we have the first term.

If  $\lambda_0 \neq 0$  and  $\lambda_2 = 2k - 1$ , we must have  $\lambda_0 + \lambda_1 = k - 1$ , and  $\lambda_0 > 0$ . So,  $1 \leq \lambda_0 \leq \frac{k-1}{2}$ , which verifies the second term.

Now, using the Lemma 4.15 and (17), we can check case by case ( $k \bmod 4$ ) that

$$N_3(k) - N_3(k-1) = \frac{k(k+1)(2k+1)}{3}.$$

Since it is easy to verify (21) for small values of  $k$ , this proves the theorem.  $\square$

**Corollary 4.17.** *Conjecture 1.1 holds for  $d = 1, 2$  and 3.*

## 5. DEGREE 4

Let  $d = 4$ . Types  $(1, 1, 1, 1)$ ,  $(3, 1)$ ,  $(1, 3)$ ,  $(2, 1, 1)$  and  $(1, 1, 2)$  are treated in Section 4. The remaining types are  $(1, 2, 1)$  and  $(2, 2)$ . In these types, positive dimensional torus fixed loci can occur.

**Example 5.1.** We give an example of a positive dimensional  $T$ -fixed locus in degree 4 of type  $(1, 2, 1)$  when  $k = 2$ .

Let  $\mathcal{F}_0 = \mathcal{O}_{\mathbb{P}^1}$ ,  $\mathcal{F}_1 = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  and  $\mathcal{F}_2 = \mathcal{O}_{\mathbb{P}^1}(-1)$ . The  $\pi_* \mathcal{O}_Y$ -module structure is

$$\phi_0 = \begin{pmatrix} x \\ y \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1} \rightarrow (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(2),$$

$$\phi_1 = \begin{pmatrix} c_1 y^2 & c_2 xy \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(2),$$

where  $c_1$  and  $c_2$  are in  $\mathbb{C}$ . It is easy to see that this satisfies the condition (5) for an equivariant sheaf. As only scaling isomorphisms are allowed, we cannot set all coefficients to be 1 using isomorphisms.

Let  $\mathcal{F}(c_1, c_2)$  be such a sheaf. One can see that  $(c_1, c_2)$  can not be  $(0, 0)$  and that  $\mathcal{F}(c_1, c_2) \simeq \mathcal{F}(\lambda c_1, \lambda c_2)$  for  $\lambda \in \mathbb{C}^*$ . So, this  $T$ -fixed locus is isomorphic to  $\mathbb{P}^1$ .

Let the  $\pi_*\mathcal{O}_Y$ -module structure of a sheaf of type  $(1,2,1)$  be

$$\phi_0 = (\alpha_1, \alpha_2)^t: \mathcal{O}_{\mathbb{P}^1}(a) \rightarrow (\mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2)) \otimes \mathcal{O}_{\mathbb{P}^1}(k)$$

$$\phi_1 = (\beta_1, \beta_2): \mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(c) \otimes \mathcal{O}_{\mathbb{P}^1}(k),$$

where  $\alpha_i$  and  $\beta_i$  are monomials with coefficient 1.

$\chi(\mathcal{F}) = 1$  is equivalent to

$$a + b_1 + b_2 + c = -3. \quad (22)$$

Without loss of generality, we assume  $b_1 \geq b_2$ . Suppose that all entries  $\alpha_i, \beta_i$  are nonzero. Then by condition (5),

$$\text{wt}(\alpha_1) - \text{wt}(\alpha_2) = \text{wt}(\beta_2) - \text{wt}(\beta_1), \quad (23)$$

where  $\text{wt}(-)$  denotes the  $T$ -weight of a monomial.

**Proposition 5.2.** *Suppose  $\mathcal{F}$  is a sheaf of type  $(1,2,1)$  as above. Assume  $b_1 \geq b_2$ . Then  $\mathcal{F}$  is stable if and only if*

- (1) *No more than one of  $\alpha_1, \alpha_2, \beta_1$  or  $\beta_2$  is zero.*
- (2)  *$c \leq -1, a \geq 0, b_1 + c \leq -2$ .*
- (3)  *$\deg(\gcd(\alpha_1, \alpha_2)) \leq b_1 + b_2 - a + k$ ,  
 $\deg(\gcd(\beta_1, \beta_2)) \leq c + k - b_1 - b_2 - 1$ .*
- (4) *If  $\alpha_1\beta_1 + \alpha_2\beta_2 = 0$ , then  $\deg(\gcd(\beta_1, \beta_2)) \leq c + k - b_1 - b_2 - a - 2$ .*

*Proof.* If at least two of  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  are zero,  $\mathcal{F}$  is decomposable.

Suppose  $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2)$  is a  $\pi_*\mathcal{O}_Y$ -submodule, where  $\mathcal{G}_0 \subset \mathcal{O}_{\mathbb{P}^1}(a)$ ,  $\mathcal{G}_1 \subset \mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2)$  and  $\mathcal{G}_2 \subset \mathcal{O}_{\mathbb{P}^1}(c)$ . Let

$$\text{rank}(\mathcal{G}) = (\text{rank}(\mathcal{G}_0), \text{rank}(\mathcal{G}_1), \text{rank}(\mathcal{G}_2)).$$

For each possible choice of the rank of  $\mathcal{G}$ , we examine the stability condition.

- (1)  $\text{rank}(\mathcal{G}) = (0, 0, 1) : c \leq -1$ .
- (2)  $\text{rank}(\mathcal{G}) = (0, 2, 1) : b_1 + b_2 + c \leq -3$  or  $a \geq 0$  by (22).
- (3)  $\text{rank}(\mathcal{G}) = (0, 1, 1) : \text{Since the degree of } \mathcal{G}_1 \text{ is no more than } b_1 \text{ as } b_1 \geq b_2, \text{ we have } b_1 + c \leq -2$ .
- (4)  $\text{rank}(\mathcal{G}) = (1, 1, 1) : \text{We can reduce to the case when } \mathcal{F}/\mathcal{G} \text{ is the torsion free cokernel of } \phi_0. \text{ So, by Lemma 4.5, stability condition is}$

$$b_1 + b_2 - a + k - \deg(\gcd(\alpha_1, \alpha_2)) \geq 0.$$

- (5)  $\text{rank}(\mathcal{G}) = (0, 1, 0) : \text{The kernel of } \phi_1 \text{ has degree } b_1 + b_2 - c - k + \deg(\gcd(\beta_1, \beta_2)). \text{ So,}$

$$\deg(\gcd(\beta_1, \beta_2)) \leq c + k - b_1 - b_2 - 1.$$

- (6)  $\text{rank}(\mathcal{G}) = (1, 1, 0)$  : A subsheaf of this type exists only if the image of  $\phi_0$  is in the kernel of  $\phi_1$ , i.e., if  $\alpha_1\beta_1 + \alpha_2\beta_2 = 0$ . In such a case, we take  $\mathcal{G}_0 = \mathcal{O}_{\mathbb{P}^1}(a)$  and  $\mathcal{G}_1 = \ker\phi_1$ . So,

$$a + (b_1 + b_2 - c - k + \deg(\gcd(\beta_1, \beta_2))) \leq -2,$$

which is the condition (4). □

Let the  $\pi_*\mathcal{O}_Y$ -module structure of a sheaf  $\mathcal{F}$  of type  $(2, 2)$  be

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \rightarrow (\mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2)) \otimes \mathcal{O}_{\mathbb{P}^1}(k).$$

$\chi(\mathcal{F}) = 1$  is equivalent to

$$a_1 + a_2 + b_1 + b_2 = -3. \quad (24)$$

Suppose that all entries  $\phi_{ij}$  are nonzero. Then by condition (5),

$$\text{wt}(\phi_{11}) - \text{wt}(\phi_{21}) = \text{wt}(\phi_{12}) - \text{wt}(\phi_{22}), \quad (25)$$

which means  $\phi_{11}\phi_{22}$  and  $\phi_{12}\phi_{21}$  are proportional.

**Proposition 5.3.** *Suppose  $\mathcal{F}$  is a sheaf of type  $(2, 2)$  as above. Assume  $a_1 \geq a_2$  and  $b_1 \geq b_2$ . Then  $\mathcal{F}$  is stable if and only if*

- (1)  $\phi_{21}$  is nonzero. No more than one of  $\phi_{ij}$  is zero.
- (2)  $a_1 \geq a_2 \geq 0$  and  $b_2 \leq b_1 \leq -1$ .
- (3)  $\deg(\gcd(\phi_{11}, \phi_{21})) \leq a_2 + b_1 + b_2 - a_1 + k + 1$ ,  
 $\deg(\gcd(\phi_{21}, \phi_{22})) \leq b_2 - b_1 - a_1 - a_2 + k - 2$ .
- (4) If  $\phi_{11}\phi_{22} = \phi_{12}\phi_{21}$ , then  
 $\deg(\gcd(\phi_{11}, \phi_{21})) \leq b_1 + b_2 - a_1 + k$ , and  
 $\deg(\gcd(\phi_{11}, \phi_{12})) \leq b_1 + k - a_1 - a_2 - 1$ .

*Proof.* If at least two of  $\phi_{ij}$  are zero,  $\mathcal{F}$  is decomposable.

Let

$$\begin{aligned} r_1 &= \deg(\gcd(\phi_{11}, \phi_{12})), & r_2 &= \deg(\gcd(\phi_{21}, \phi_{22})), \\ s_1 &= \deg(\gcd(\phi_{11}, \phi_{21})), & s_2 &= \deg(\gcd(\phi_{12}, \phi_{22})). \end{aligned}$$

Then by (25),

$$r_2 = r_1 + b_2 - b_1 \text{ and } s_2 = s_1 + a_1 - a_2, \quad (26)$$

provided that  $\phi_{ij}$  are all nonzero.

Suppose  $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$  is a  $\pi_*\mathcal{O}_Y$ -submodule. For each possible choice of the rank of  $\mathcal{G}$ , we examine the stability conditions.

- (1)  $\text{rank}(\mathcal{G}) = (0, 1)$  :  $b_1 \leq -1$ .
- (2)  $\text{rank}(\mathcal{G}) = (0, 2)$  :  $b_1 + b_2 \leq -2$  which is implied by the above condition (1).

- (3)  $\text{rank}(\mathcal{G}) = (1, 2) : a_2 \geq 0$ .  
 (4)  $\text{rank}(\mathcal{G}) = (1, 1) : \text{Let } \mathcal{G}_0 = \mathcal{O}_{\mathbb{P}^1}(m) \text{ and } \mathcal{G}_1 = \mathcal{O}_{\mathbb{P}^1}(n). \text{ If } a_2 < m \leq a_1, \mathcal{G}_0 \text{ is a subsheaf of } \mathcal{O}_{\mathbb{P}^1}(a_1). \text{ Hence, we can replace } \mathcal{G}_0 \text{ by } \mathcal{O}_{\mathbb{P}^1}(a_1) \text{ and take } \mathcal{G}_1 \text{ to be the saturation of the image of } \mathcal{O}_{\mathbb{P}^1}(a_1) \text{ under } \phi. \text{ The quotient is } (\mathcal{O}_{\mathbb{P}^1}(a_2), \text{ the torsion free cokernel of } \phi|_{\mathcal{O}_{\mathbb{P}^1}(a_1)}). \text{ So, for } \mathcal{F} \text{ to be stable, we must have}$

$$a_2 + b_1 + b_2 - a_1 + k - s_1 \geq -1,$$

by Lemma 4.5. Note that if  $\phi_{21}$  is zero,  $s_1 = b_1 - a_1 + k$ . Then the quotient has degree  $a_2 + b_2 \leq -2$  by (24) contradicting the stability.

Now suppose  $m \leq a_2$ . If  $n \leq b_2$ , since  $a_2 + b_2 \leq -2$ , there is nothing to check. If  $b_2 \leq n \leq b_1$ , we can replace  $\mathcal{G}_1$  by  $\mathcal{O}_{\mathbb{P}^1}(b_1)$  and take  $\mathcal{G}_0$  to be the inverse image of  $\mathcal{O}_{\mathbb{P}^1}(b_1)$ , i.e., the kernel of the map

$$(\phi_{21}, \phi_{22}) : \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(b_2) \otimes \mathcal{O}_{\mathbb{P}^1}(k).$$

Then the condition is

$$b_1 + a_1 + a_2 - b_2 - k + r_2 \leq -2.$$

- (5)  $\text{rank}(\mathcal{G}) = (1, 0) \text{ or } (2, 1) : \text{A subsheaf of these types exists only if the image of } \phi \text{ has rank 1, in other words, if } \phi_{11}\phi_{22} = \phi_{12}\phi_{21}. \text{ Then, the torsion free cokernel of } \phi \text{ has degree}$

$$b_1 + b_2 - a_1 + k - s_1 = b_1 + b_2 - a_2 + k - s_2,$$

and the kernel of  $\phi$  has degree

$$a_1 + a_2 - b_1 - k + r_1 = a_1 + a_2 - b_2 - k + r_2,$$

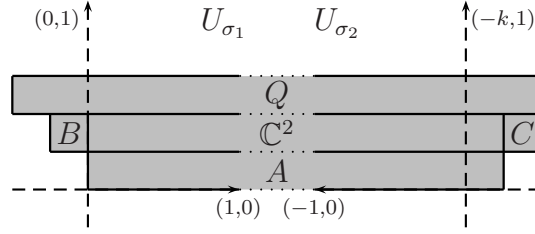
by (26). Hence the conditions are

$$s_1 \leq b_1 + b_2 - a_1 + k \text{ and } r_1 \leq b_1 + k - a_1 - a_2 - 1.$$

□

*Remark 5.4.* As we will see in the next example, all positive dimensional loci of type  $(1, 2, 1)$  can be expressed as a GIT quotient of  $(\mathbb{P}^1)^4$  by the action of  $SL_2$ . While the linearization may be different, the quotient is always isomorphic to  $\mathbb{P}^1$ . Similar argument for type  $(2, 2)$  holds. So, we can see that all  $T$ -fixed loci in degree 4 are either isolated points or  $\mathbb{P}^1$ .

**Example 5.5.** In Example 5.1,  $\mathcal{F}_0$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are unchanged along the one dimensional torus fixed locus. The condition (4) in Propositions 5.2 and 5.3 suggests this is not true in general.

FIGURE 3. Sheaf of type  $(1, 2, 1)$ 

Assume  $k = 3$  and let  $\mathcal{F}_0 = \mathcal{O}_{\mathbb{P}^1}(1)$ ,  $\mathcal{F}_1 = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  and  $\mathcal{F}_2 = \mathcal{O}_{\mathbb{P}^1}(-2)$ . The  $\pi_*\mathcal{O}_Y$ -module structure is

$$\phi_0 = \begin{pmatrix} x \\ y \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(3),$$

$$\phi_1 = \begin{pmatrix} c_1xy & c_2x^2 \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(3),$$

where  $c_1$  and  $c_2$  are in  $\mathbb{C}$ . By Proposition 5.2, we can check the corresponding sheaf  $\mathcal{F}(c_1, c_2)$  is stable unless  $c_1 = -c_2$ .

As the  $T$ -fixed locus  $M_d(k)$  is compact, the limit of above family at  $c_1 = -c_2$  exists in  $M_d(k)^T$ . To see what the limit is, we need to examine  $\Delta$ -family described in Proposition 3.1.

Assume the fixed point in the open set  $U_{\sigma_1}$  is given by  $x = 0$  and the fixed point in  $U_{\sigma_2}$  by  $y = 0$ . Then the above  $\pi_*\mathcal{O}_Y$ -module structure has weight space decomposition as Figure 3.

In Figure 3,  $A, B, C$  and  $Q$  are one dimensional. By Proposition 3.1,  $T$ -fixed sheaves with such weight space decomposition are determined by inclusions of  $A, B$  and  $C$  into  $\mathbb{C}^2$  and a surjection  $\mathbb{C}^2 \rightarrow Q$ . The  $SL_2(\mathbb{C})$  action on  $\mathbb{C}^2$  via change of basis encodes isomorphism between sheaves. See [12, Chapter 3] for a detailed discussion.

We identify  $\mathbb{C}^2 \rightarrow Q$  with its kernel  $K$  so that  $A, B, C$  and  $K$  are in  $Gr(1, \mathbb{C}^2) \simeq \mathbb{P}^1$ . We want to relate Gieseker stability to GIT stability condition for the action of  $SL_2(\mathbb{C})$  on  $(\mathbb{P}^1)^4$ . It can be checked that the associated sheaf is Gieseker stable unless

$$A = B \text{ or } A = C \text{ or } A = K \text{ or } B = C = K. \quad (27)$$

Meanwhile, a point  $(p_1, p_2, p_3, p_4) \in (\mathbb{P}^1)^4$  is GIT stable with respect to a line bundle  $\mathcal{O}(k_1, k_2, k_3, k_4)$  if and only if for any point  $p \in \mathbb{P}^1$

$$\sum_{p=p_i} k_i < \frac{1}{2} \sum_{i=1}^4 k_i. \quad (28)$$

See [4, Theorem 11.4], [15, Section 4.4]. If we take  $k_1 = 2, k_2 = k_3 = k_4 = 1$ , these two conditions agree with each other. This is an

example of matching GIT stability and Gieseker stability discussed in [12, Chapter 3].

Therefore, the  $T$ -fixed locus is

$$(\mathbb{P}^1)^4 // SL_2(\mathbb{C}) \simeq \mathbb{P}^1.$$

The condition  $c_1 = -c_2$  is equivalent to  $A = K$ . It is easy to check that at the limit in  $(\mathbb{P}^1)^4 // SL_2(\mathbb{C})$ , we have  $B = C$  and  $A, B, K$  are distinct. By reading equivariant vector bundles in each rows, we can see the limit has  $\pi_* \mathcal{O}_Y$ -module structure

$$\phi_0 = \begin{pmatrix} xy \\ 1 \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \otimes \mathcal{O}_{\mathbb{P}^1}(3),$$

$$\phi_1 = \begin{pmatrix} x & x^2y \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(3).$$

Note that since  $xy$  is a multiple of 1, or  $x^2y$  is a multiple of  $x$ , we can set all the coefficients of monomials to be 1 up to isomorphism.

*Remark 5.6.* Conjecture 1.1 combined with the Gromov-Witten theory predicts that

$$n_4(k) = -\frac{k(k+1)^2(k+2)(2k^2+4k+1)}{12}. \quad (29)$$

Based on the classification of  $T$ -equivariant stable sheaves studied above and in Section 4, we can compute  $N_4(k)$ . The author has checked that the result is consistent with (29) when  $k \leq 100$  with the help of a computer. However, we don't have a proof for general  $k$ .

## 6. EQUIVARIANT RESIDUE

The virtual tangent space at  $\mathcal{F} \in M_d(k)$  is

$$\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{F}) - \mathrm{Ext}_X^2(\mathcal{F}, \mathcal{F}).$$

Since  $T$  preserves a canonical Calabi-Yau form, the canonical bundle on  $X$  is trivial with trivial weight. By equivariant Serre duality,

$$\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{F}) \simeq \mathrm{Ext}_X^2(\mathcal{F}, \mathcal{F})^*$$

as  $T$ -representations. So, the dual weights of the moving parts will be canceled and we just count signs.

Let  $\mathcal{H}_k$  be the Hirzebruch surface whose toric fan has ray generators  $u_1 = (-1, k)$ ,  $u_2 = (0, 1)$ ,  $u_3 = (1, 0)$ ,  $u_4 = (0, -1)$ . Denote the corresponding divisors by  $D_1, D_2, D_3, D_4$ .

The total space  $Y$  of  $\mathcal{O}_{\mathbb{P}^1}(k)$  can be described as a toric variety by the fan  $\{\mathrm{Cone}(u_3, u_4), \mathrm{Cone}(u_4, u_1)\}$ . Hence,  $Y$  is a subvariety of  $\mathcal{H}_k$  and the zero section of  $Y$  is the divisor  $D_4$ . Let  $i: Y \rightarrow \mathcal{H}_k$  be the inclusion.



The anticanonical class of  $\mathcal{H}_k$

$$-K = (D_1 + D_2 + D_3 + D_4) = 2D_2 + (k+1)D_3$$

is ample. so we have a well defined moduli space

$$M_{\mathcal{H}_k}(d) = \{\mathcal{F} \text{ sheaf on } \mathcal{H}_k: c_1(\mathcal{F}) = dD_4, \chi(\mathcal{F}) = 1, (-K)\text{-(semi)stable}\}.$$

Recall that  $L$  is the pullback of  $\mathcal{O}_{\mathbb{P}^1}(1)$  to  $Y$ . Then,  $i^*\mathcal{O}(-K) \simeq L^{\otimes(k+1)}$  and we have

$$\chi(\mathcal{F} \otimes L^{\otimes n(k+1)}) = \chi(i_*\mathcal{F} \otimes \mathcal{O}(-K)^{\otimes n}).$$

Thus, since  $k+1 \geq 0$ ,  $i_*\mathcal{F}$  is  $(-K)$ -semistable if and only if  $\mathcal{F}$  is  $L$ -semistable. Hence,  $i_*$  induce an injective morphism from  $M_d(k)$  to  $M_{\mathcal{H}_k}(d)$ .

**Proposition 6.1.**  *$M_{\mathcal{H}_k}(d)$  is a smooth projective variety of dimension  $kd^2 + 1$ .*

*Proof.* Any semistable sheaf  $\mathcal{F}$  in  $M_{\mathcal{H}_k}(d)$  is necessarily stable because  $\chi(\mathcal{F}) = 1$ . So  $M_{\mathcal{H}_k}(d)$  is projective variety. By Serre duality

$$\text{Ext}^2(\mathcal{F}, \mathcal{F}) = \text{Hom}(\mathcal{F}, \mathcal{F} \otimes K)^\vee = 0,$$

since  $\mathcal{F}$  is  $(-K)$ -stable. Therefore, there is no obstruction and  $M_{\mathcal{H}_k}(d)$  is smooth.

We compute  $\dim \text{Ext}^1(\mathcal{F}, \mathcal{F})$  using Riemann-Roch.

$$\chi(\mathcal{F}, \mathcal{F}) = 1 - \dim \text{Ext}^1(\mathcal{F}, \mathcal{F}) = \int_{\mathcal{H}_k} \text{ch}^\vee(\mathcal{F}) \text{ch}(\mathcal{F}) \text{td}(\mathcal{H}_k)$$

Since the rank of  $\mathcal{F}$  is zero and  $c_1(\mathcal{F}) = dD_4$ , the degree 2 term of right side is  $-d^2 D_4^2 = -kd^2$ . Therefore,

$$\dim \text{Ext}^1(\mathcal{F}, \mathcal{F}) = 1 - \chi(\mathcal{F}, \mathcal{F}) = kd^2 + 1.$$

Thus  $\dim M_{\mathcal{H}_k}(d) = kd^2 + 1$ . □

**Corollary 6.2.**

$$n_d(k) = (-1)^{kd^2+1} e_{\text{top}}(M_d(k))$$

*Proof.*  $M_d(k)$  is open subscheme of  $M_{\mathcal{H}_k}(d)$ , hence smooth of dimension  $kd^2 + 1$ . Then, this is a consequence of general properties of Donaldson-Thomas type invariant with symmetric obstruction theory [1]. □

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